$$\int_a^b \underset{\substack{\downarrow \\ \text{συνx}}}{f}(\varphi(t))\,\underbrace{\varphi'(t)}_{\text{ολοκλ.}}\,dt = \int_{\varphi(a)}^{\varphi(b)} f(s)\,ds$$

$\exists \varphi'$

φ συνx

$$\begin{tikzcd}
{[a,b]} \arrow[r, "\varphi"] \arrow[dr, "(f\circ\varphi)(t)"'] & {[\gamma,\delta]} \arrow[d, "f"] \\
 & \mathbb{R}
\end{tikzcd}$$

$$F(x) = \int_\gamma^x f$$

$$F'(x) = f(x)$$

$$f(\varphi(t))\varphi'(t) = F'(\varphi(t))\varphi'(t) = [F(\varphi(t))]' = [F\circ\varphi]'(t)$$

$$I = \int_a^b [F\circ\varphi]'\,dt = (F\circ\varphi)(b) - (F\circ\varphi)(a) = F(\varphi(b)) - F(\varphi(a))$$

$$= \int_{\varphi(a)}^{\varphi(b)} f(u)\,du = J$$

$$\frac{1}{2}\int_{x=0}^{\pi/2} [2x\cos(x^2)]\,dx = \frac{1}{2}[\sin(x^2)]_0^{\pi/2} = \dots$$

$$\int_{x=0}^{\pi/2} \cos(x)\,dx = \int_0^{\pi/2} [\sin(x)]'\,dx = [\sin x]_0^{\pi/2}$$

2ο Θ. αλλαγ. μεταβλ. $\quad \psi:[a,b]\to\mathbb{R}$, $\exists\ \psi'$ συν στο $[a,b]$

$\psi'(t)\neq 0 \quad \forall t\in[a,b]$

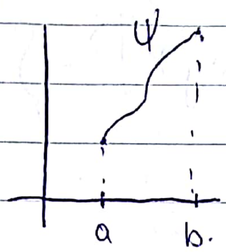
$$\overset{I_1}{\int_a^b f(\psi(t))\,dt} = \int_{\psi(a)}^{\psi(b)} f(u)[\psi^{-1}(u)]'\,du = I_2$$

$\psi(t)=u \Rightarrow t=\psi^{-1}(u) \qquad dt=[\psi^{-1}(u)]'\,du$

$$I_2=\int_{\psi(a)}^{\psi(b)} f(u)[\psi^{-1}(u)]'\,du$$

$\psi'(t_1)>0$

$\psi'(t_2)<0$



1. ψ γν. αυξ
   ψ συνεχ $\Rightarrow \exists\ \psi^{-1}$: συνεχ
2. ψ γν αυξ
   ψ παραγωγ $\Rightarrow \exists\ \psi^{-1}$ παραγ

$\psi(\psi^{-1}(x))=x, \quad \forall x\in I$

$\psi^{-1}(\psi(s))=s \quad \forall s$

$\psi'(\psi^{-1}(x))\cdot[\psi^{-1}(x)]'=1 \Rightarrow$

$$(\psi^{-1}(x))'=\frac{1}{\psi'(\psi^{-1}(x))}$$

$$[a,b] \underset{\psi^{-1}}{\leftrightarrows} [\psi(a),\psi(b)]=I$$

$$\int_{\psi(a)}^{\psi(b)} f(\psi(\psi^{-1}(u)))[\psi^{-1}(u)]'\,du = \int_{\psi(a)}^{\psi(b)} \underbrace{(f\circ\psi)}_{G}(\psi^{-1}(u))[\psi^{-1}(u)]'\,du$$

$$\int_{\psi(a)}^{\psi(b)} G(\psi^{-1}(u))[\psi^{-1}(u)]'\,du = \int_{s=\psi^{-1}(\psi(a))}^{\psi^{-1}(\psi(b))} G(s)\,ds = \int_a^b G(s)\,ds$$

Θεώρημα: $f, g : [a,b] \to \mathbb{R}$ $\exists f', g'$ R-ολοκληρώσιμες.

$$\int_a^b f(x) g'(x) dx = [f(b)g(b) - f(a)g(a)] - \int_a^b f'(x) g(x) dx$$

$$\int_a^b [fg' + f'g] = f(b)g(b) - f(a)g(a)$$

$$\int_a^b [f(x)g(x)]' dx = [f(x)g(x)]_a^b$$

$$\int_2^3 x' \log(x) dx = [x \log x]_2^3 - \int_2^3 x (\log(x))' dx$$

$$\| $$

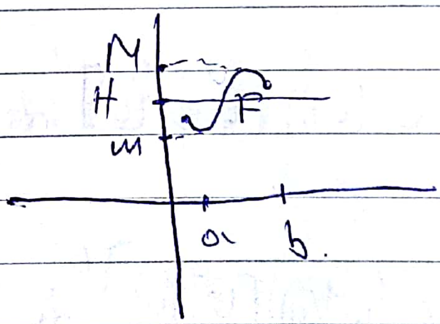
$$\int_2^3 \log(x) dx$$

Πρόταση: $f : [a,b] \to \mathbb{R}$ $g : [a,b] \to \mathbb{R}$ R-ολοκληρώσιμη
συνεχής
και g διατηρεί σταθερό πρόσημο

$\Rightarrow \exists \xi \in [a,b]$ ώστε $\int_a^b f(x) g(x) dx = f(\xi) \int_a^b g(x) dx$

$f : [a,b] \to \mathbb{R}$ $\exists \xi \in [a,b]$

$$f(\xi) = \frac{1}{b-a} \int_a^b f = H$$



$$m \le H \le \frac{1}{b-a} \int_a^b M dx = M \Rightarrow \exists \xi \in [a,b] : f(\xi) = H$$

$g(x) \geq 0 \quad \forall x$ $\quad \exists \ m = \min_{[a,b]} f$

ή $\quad m \leq M$

$g(x) \leq 0 \quad \forall x \in [a,b]$ $\quad \exists \ M = \max_{[a,b]} f$

και $m \leq f(x) \leq M \quad \forall x \in [a,b]$

χ.β.τ.γ $g(x) \geq 0 \quad \forall x \in [a,b]$

$\Rightarrow m g(x) \leq f(x) g(x) \leq M g(x) \quad \forall x \in [a,b]$

$$m \int_a^b g \leq \int_a^b fg \leq M \int_a^b g \quad (1)$$

$g(x) \geq 0 \quad \forall x$

$\int_a^b g \geq 0$ $\quad$ Διακρίνουμε περιπτώσεις

i) $\int_a^b g = 0$ $\Rightarrow$ διαλέγουμε τυχαίο $\xi \in [a,b]$

ii) $\int_a^b g > 0$

$$(1) \Rightarrow \underset{\min f}{m} \leq \boxed{\frac{\int_a^b fg}{\int_a^b g}} \leq M = \max f$$

f συνχ $\Rightarrow \exists \ \xi \in [a,b] : f(\xi) = H$

Άσκηση: $f, g : [a,b] \to \mathbb{R}$, f συνχ στο $[a,b]$

g μονότονη, $\exists g'$ στο $[a,b]$, g' συνχ στο $[a,b]$

$\Rightarrow \exists \ \xi \in [a,b]$:

$$\int_a^b fg = g(a) \int_a^\xi f + g(b) \int_\xi^b f$$

Ορίζουμε: $F : [a,b] \to \mathbb{R}$ $\quad F(x) = \int_a^x f$ $\quad f$ συνχ $\Rightarrow \exists \ F'(x) = f(x)$

$\forall x \in [a,b]$ $\quad \xi = ;$, $\int_a^\xi f = F(\xi)$

Αρκεί να δ.ο: $\int_a^b F'g = g(a)F(\xi) + g(b)(F(b)-F(\xi))$

για κάποιο $\xi \in [a,b]$

$I = [F(x)g(x)]_a^b - \int_a^b F(x)g'(x)dx = F(b)g(b) - \int_a^b Fg' =$

$F(b)g(b) - F(a)g(a) - \int Fg' = -F(\xi)\int_a^b g'(x)dx +$

$F(b)g(b)$

Ασκ 2) $f:[0,1] \to \mathbb{R}$ συνεχ: $\int_0^x f = \int_x^1 f \quad \forall x \in [0,1] \Rightarrow f(x)=0 \quad \forall x$

$F(x) = \int_0^x f, \quad x \in [0,1] \qquad (1) \Leftrightarrow F(x) = \int_0^1 f - F(x)$

$F'(x) = f \qquad F(x) = \int_0^1 f/2$

$\Rightarrow F'(x) = 0 \quad \forall x$

$\quad \| \qquad\qquad f \equiv 0$

$f(x)$

Ασκ 3) $f, h : [0,+\infty) \to [0,+\infty)$, h συνεχ $\exists f'$. $F(x) = \int_0^{f(x)} h(t)dt$

ΝΟ η F είναι παρ/μη $\Rightarrow F' = ;$

$G(x) = \int_0^x h(t)dt \qquad G:[0,+\infty) \to \mathbb{R} \qquad G' = h$

Άρα $F(x) = G(f(x)) = (G \circ f)(x)$

$\exists f'$, $F'(x) = [G(f(x))]' = G'(f(x)) \cdot f'(x)$

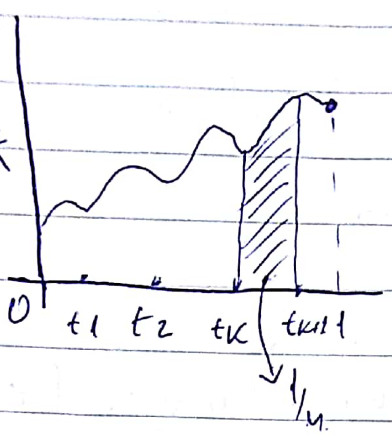
$= h(f(x)) f'(x)$

Ασκ 1) $f: [0,1] \to \mathbb{R}$ R-ολοκληρ.
$f(x) \geq 0 \ \forall x \in [0,1]$ με $\int_0^1 f = 1$

και $n \in \mathbb{N} \to \exists$ διαμέριση $P = \{t_0 = 0 < t_1 < \dots < t_{n-1} < t_n = 1\}$ ώστε $\int_{t_k}^{t_{k+1}} f = \frac{1}{n} \ \forall k = 0, 1, \dots, n-1$

$\int_0^{t_1} f = \frac{1}{n}$

$\int_0^{t_2} f = \frac{2}{n}$



$\int_0^{t_k} f = \frac{k}{n} \quad \forall k = 0, \dots, n-1$

Αν ① αληθεύει

② $\int_{t_k}^{t_{k+1}} f = \int_0^{t_{k+1}} f - \int_0^{t_k} f = \frac{k+1}{n} - \frac{k}{n} = \frac{1}{n}$ χ

$f(x) \geq 0 \ \forall x \Rightarrow$ ② $t_k < t_{k+1}$

$\int_2^1 g(x)dx = -\int_1^2 g \leq 0 \qquad g(x) \geq 0 \ \forall x$

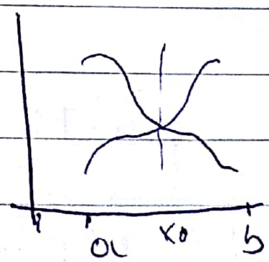
Αν βρούμε $t_k, \ k = 0, 1, \dots, n-1, n$ $\int_0^{t_k} f = \frac{k}{n}$

F συνεχ
$F(0) = 0 \quad F(1) = 1$
$0 < \frac{k}{n} < 1$

$\Big\} \ \exists t_k \in (0,1) : F(t_k) = k/n$

$f, g : [a,b] \to \mathbb{R}$ συνεχείς.

$\int_a^b f = \int_a^b g \overset{\Delta.\Theta}{\Rightarrow} \exists x_0 \in [a,b] : f(x_0) = g(x_0)$



$h : [a,b] \to \mathbb{R} \qquad h(x) = f(x) - g(x)$

$$\int_a^b h = \int_a^b f - \int_a^b g = 0$$

h συνx | ψαχνουμε $x_0 \in [a,b] : h(x_0) = 0$

$\exists \xi \in [a,b]$ $(b-a)\, h(\xi) = \int_a^b h$ (h συνx)

ΘΜΤ $\Downarrow$ $\quad \| \quad 0.$

$\exists \xi = x_0 \quad h(x_0) = 0$

$\int_a^b h = 0 \Rightarrow \exists x_0 \in [a,b] : h(x_0) = 0$

h συνx.

Εστω ότι $\not\exists x_0 \quad h(x_0) = 0 \Rightarrow h(x) > 0 \ \forall x \in [a,b]$ ①.

$h(x) < 0 \ \forall x$ ②

① άτοπο, $\left.\begin{matrix} h(x) > 0 \\ h \text{ συνx} \end{matrix}\right\} \xRightarrow{\text{ασκ}} \int_a^b h > 0$

② $\Rightarrow$ άτοπο,

$\left.\begin{matrix} h(x) < 0 \ \forall x \\ h \text{ συνx} \end{matrix}\right\} \Rightarrow \int_a^b h < 0$ (άτοπο $\int_a^b h = 0$)

$$\int_{t=a}^{b} f(\varphi(t))dt = \int_{\varphi(a)}^{\varphi(b)} f(u)[\varphi^{-1}(u)]' du \qquad \varphi'(t) \neq 0 \ \forall t \in [a,b]$$

$\varphi'(t) \neq 0$, $\exists \varphi^{-1}$ $\qquad \varphi(t) = u \Leftrightarrow t = \varphi^{-1}(u)$

$dt = (\varphi^{-1}(u))' du$

$\varphi, \varphi' \neq 0$

$\varphi' \rightsquigarrow (\varphi^{-1})'(x)$ | $I = \int \frac{\cos(\sqrt{x})}{\sqrt{x}} dx$

$\varphi \nearrow \downarrow$

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$= \int f(x)dx = F(x)$ = παράγουσα της f

$\int F'(x)dx$ $\quad F'(x) = f(x)$

$= F(x) + c$

$$f(x) = \frac{\cos(\sqrt{x})}{\sqrt{x}}$$

$$I = \int 2\frac{\cos(\sqrt{x})}{2\sqrt{x}}dx = \cancel{\int f(x)dx = \text{παράγουσα της } f}$$

$$= 2\int (\sqrt{x})' \cdot \cos(\sqrt{x})dx = 2\int (\sin(\sqrt{x}))'dx = 2\sin(\sqrt{x}) + 2c$$

$$\int_{t=a}^{b} f(\varphi(t))dt = \int_{\varphi(a)}^{\varphi(b)} f(u)[\varphi^{-1}(u)]'du$$

$u = \varphi(t) = \sqrt{t}$

$\int f(\sqrt{t})dt =$

$\sqrt{x} = u$

$x = u^2$

$dx = 2u\,du.$

$$I = \int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$$

$$= \int \frac{\cos(u)}{u} \cdot 2u\,du.$$

$$= 2\int \cos(u)du = 2\sin u + c = 2\sin\sqrt{x} + c$$

$x \to \varphi(x) = u.$

$\varphi: [a,b] \to [\varphi(a), \varphi(b)]$

$\varphi'(t) > 0 \;\forall t$

$\varphi^{-1}: [\varphi(a), \varphi(b)] \to [a,b].$

$u \to \varphi^{-1}(u) = x$

$$[\varphi^{-1}(u)]' = \frac{1}{\varphi'(\underbrace{\varphi^{-1}(u)}_{x})}$$

$$\Leftrightarrow [\varphi^{-1}(u)]' \cdot (\varphi'(\varphi^{-1}(u)))' = 1$$

$= u'$

$\psi : (0,1) \to (0,1) \qquad \psi(x) = \sqrt{x} \Rightarrow \psi^{-1}(u) = u^2$

$\sqrt{x} = u \Rightarrow x = u^2$

1) $[\psi^{-1}(u)]' = 2u$ $\qquad$ 2) $[\psi^{-1}(u)]' = \dfrac{1}{\psi'(\psi^{-1}(u))} =$

$= \dfrac{1}{\psi'(u^2)} = \dfrac{1}{\frac{1}{2u}} = 2u$

$\psi'(x) = \dfrac{1}{2\sqrt{x}} \Rightarrow \psi'(u^2) = \dfrac{1}{2\sqrt{u^2}} = \dfrac{1}{2u}$

Βασικοί τύποι ολοκληρωμάτων.

$\int x^a dx = \dfrac{x^{a+1}}{a+1} + c \qquad \int \dfrac{dx}{x} = (\ln|x|) + c$

$a \neq -1$

$\int e^x dx = e^x + c \qquad \int (\sin x) dx = -\cos x + c$

$\int \cos x dx = \sin x + c$

$\int \dfrac{dx}{\cos^2 x} = \tan(x) + c \qquad \int \dfrac{dx}{\sin^2 x} = -\cot(x)$

$I = \int \tan(x) dx = \int \dfrac{\sin x}{\cos x} dx = -\int (\ln(\cos x))' dx =$

$-\ln(|\cos x|) + c$

$I = \int \cos^2(x) dx = \int \dfrac{1+\cos(2x)}{2} dx$

$\cos(2x) = \cos^2(x) - \sin^2(x)$

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

$= \int \dfrac{dx}{2} + \dfrac{1}{2} \int \cos(2x) dx$

$\cos(2x) = 2\cos^2(x) - 1$

$\cos^2(x) = \dfrac{1+\cos(2x)}{2}$

$= \dfrac{x}{2} + \dfrac{1}{2}\dfrac{1}{2} \int (2x)' (\cos(2x)) dx + c$

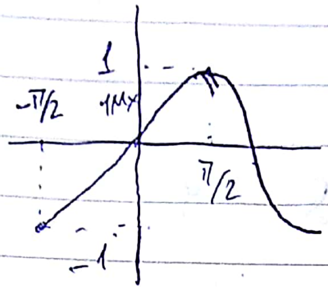
$$= \frac{x}{4} + \frac{1}{4}\sin(2x) + C$$

$$(\text{τοξημ}x)' = \frac{1}{\sqrt{1-x^2}} \qquad x \in (-1,1)$$

$\parallel$

$(\arcsin x)'$

$$\text{ημ}(x) : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1].$$



$$\text{τοξημ}(\cdot) : (-1,1) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$\parallel$

$\text{ημ}^{-1}(x).$

$$[\text{ημ}^{-1}(x)]' = [\sin^{-1}(x)]' = \frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-x^2}}$$

$$\varphi(x) = \text{ημ}x \ (\sin x), \quad \varphi'(t) = \text{συν}t \Rightarrow \varphi'(\varphi^{-1}(x)) = \cos(\sin^{-1}(x)) =$$

$$= \cos(\arcsin(x))$$

$$[\varphi^{-1}(x)]' = \frac{1}{\varphi'(\varphi^{-1}(x))}$$

$$\cos(\arcsin(x)) = \sqrt{1-x^2} \Leftrightarrow [\cos(\underbrace{\arcsin(x)}_{y})]^2 = 1-x^2$$

$$\cos^2(\arcsin(x)) + x^2 = 1$$

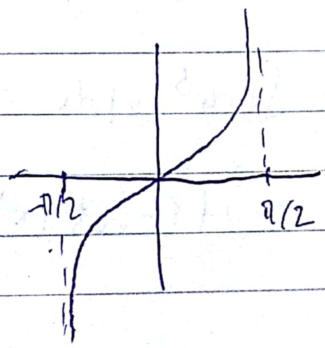
$$1 = \cos^2(\arcsin x) + \sin^2(\arcsin(x))$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + c$$

$x \in (-1,1)$

$$[\arctan(x)]' = \frac{1}{1+x^2}, \quad x \in \mathbb{R}.$$

$[\tan^{-1}(x)]$



$\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R} \qquad \tan^{-1} : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$[\tan^{-1}(x)]' = \frac{1}{\varphi'(\tan^{-1}(x))} = \frac{1}{\frac{1}{\cos^2(\tan^{-1}(x))}} = \cos^2(\tan^{-1}(x)) = \frac{1}{1+x^2}$$

$x \in \mathbb{R}$

$$[\varphi^{-1}(x)]' = \frac{1}{\varphi'(\varphi^{-1}(x))} \qquad \varphi(t) = \tan(t) \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\sin^2 y + \cos^2 y = 1$$

$$\tan^2 y + 1 = \frac{1}{\cos^2(y)} \quad \text{αληθές } \forall y.$$

$$\Rightarrow \frac{1}{\cos^2(\tan^{-1}(x))} = 1 + [\tan(\tan^{-1}(x))]^2$$

$$\Rightarrow \frac{1}{\cos^2(\tan^{-1}(x))} = 1 + x^2$$

$$e^{ia} = \cos(a) + i\sin(a)$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x.$$

$$\sin 2x = 2\cos(x) \cdot \sin(x)$$

$$\int \cos^4(x)dx = \int [\cos^2(x)]^2 dx = \int \left(\frac{1+\cos(2x)}{2}\right)^2 dx$$

$$\cos^2 x = \frac{1+\cos(2x)}{2} \quad \Big| \quad = \int \left[\frac{1}{4} + \frac{1}{4}\cos^2(2x) + \cos 2x\right] dx$$

$$= \frac{x}{4} + \frac{1}{4}\int \cos^2(2x)dx + \frac{\sin(2x)}{2} + C$$

$$I = \int \sin^5(x)dx = \int \sin^4(x) \cdot \sin(x) dx = -\int [\sin^2(x)]^2 (\cos x)' dx$$

$$= -\int [1 - \cos^2(x)]^2 (\cos x)' dx$$

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$$= -\int [1-u^2]^2 du = -\int (1+u^4-2u^2)du$$

$$= -u - \frac{u^5}{5} + \frac{2u^3}{3} + c$$

$$\int f'(x)g(x)dx = F(x)g(x) - \int f g' dx$$

$$I = \int x \log(x)\,dx = \int \left(\frac{x^2}{2}\right)' \log(x)$$

$$= \frac{x^2}{2}\log(x) - \int \frac{x^2}{2}\frac{1}{x}dx = \frac{x^2}{2}\log(x) - \int \frac{x}{2}dx =$$

$$= -\frac{x^2}{4} + c$$

$$I = \int (e^x)' \sin(x)\,dx = e^x \sin(x) - \int (e^x)' \cos x\,dx.$$

$$= e^x \sin(x) - e^x \cos x + \int e^x(-\sin x)dx \Rightarrow$$

$$\cancel{2}I = \frac{e^x \sin x - e^x \cos x}{2}$$

$$\int x\cos(x)dx = \int x(\sin x)'dx = x\sin x - \int x'\sin(x)dx.$$

$$= x\sin x - \int \sin(x)dx = x\sin(x) + \cos(x) + c$$

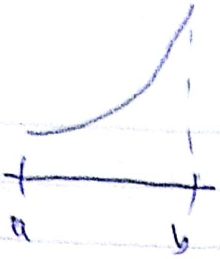
Ασκ 1) $f:[0,1] \to \mathbb{R}$ $\quad f|[b,1]$ ολοκλ $\forall b$ $\quad \exists A: |f(x)| \le A \quad \forall x \in [0,1]$

$\Rightarrow f$ ολοκλ στο $[0,1]$ $\quad$ φραγμένη στο $[0,1]$.

Επιπλέον $\int_0^1 f = \lim_{b \to 0^+} \int_b^1 f$

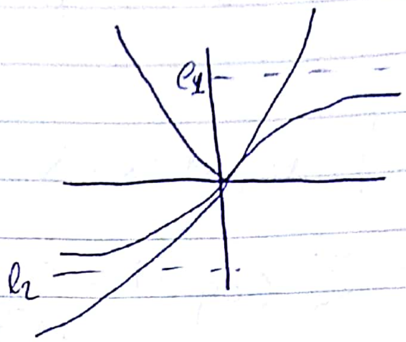
$$0 \xleftarrow{b\to 0} \left|\int_b^1 f - \int_0^1 f\right| = \left|\int_b^1 f - \left(\int_0^b f + \int_b^1 f\right)\right| = \left|-\int_0^b f\right| = \left|\int_0^b f\right|$$

$$\le \int_0^b |f(x)|dx \le A(b-0) = Ab \xrightarrow{b\to 0^+} 0$$



$\xrightarrow{f} \mathbb{R}$ συνεχ τότε η f είναι ομ συνεχ $\Leftrightarrow$ $\exists \lim_{x\to a^+} f(x)$ κ' $\exists \lim_{x \to b^-} f(x)$

$f:\mathbb{R}\to\mathbb{R}$ συνεχ, $\exists \lim_{x\to\pm\infty} f(x) = \ell_1$ $\Rightarrow$ f ομ συνεχ



$f:[0,+\infty)\to\mathbb{R}$ συνεχ, $\exists \lim_{x\to+\infty} f(x) = \ell$ $\Rightarrow f$ ομ. συνεχ

$f:\mathbb{R}\to\mathbb{R}$

$f/(-\infty,0)$, $f/[0,+\infty)$

$$(\text{τοξημ}x)' = \frac{1}{\sqrt{1-x^2}} \quad (x\in(-1,1)) \Rightarrow \int \frac{dx}{\sqrt{1-x^2}} = \text{τοξημ}(x) + c$$

$$\int \frac{dx}{1+x^2} = \text{τοξεφ}(x) + c \quad x\in\mathbb{R}$$

$$I_1 = \int \varepsilon\varphi^2(x)\,dx$$

$$\varepsilon\varphi(x) = \frac{\sin x}{\cos x} \qquad \cos^2 x + \sin^2 x = 1 \Rightarrow 1 + \varepsilon\varphi^2(x) = \frac{1}{\cos^2(x)}$$

$$\Rightarrow \varepsilon\varphi^2(x) = \frac{1}{\cos^2(x)} - 1$$

$$\int \varepsilon\varphi^2(x)\,dx = \int \frac{1}{\cos^2(x)} - \int dx = \varepsilon\varphi(x) - x + c.$$

$$I_2 = \int \sigma\varphi^2(x)\,dx$$

$$\int \frac{x^2-2x+3}{x^5-7x+8}\,dx \qquad \int \frac{P(x)}{Q(x)}\,dx$$

$P(x)$: n-βαθμ
$Q(x)$: m-βαθμ

Πολυώνυμα πραγματ. συντ.

$$\int \frac{dx}{(x-0)^1} = \ln|x| + C \qquad \int \frac{1}{(x-1)^1}dx = \ln|x-1| + C$$

$$\int \frac{1}{(x-p)^k}dx = \int (x-p)^{-k}dx = \frac{(x-p)^{-k+1}}{-k+1} + C$$

$k \neq 1$

$$\int \frac{P(x)}{Q(x)}dx$$

$P(x) \to$ m-βαθμού.

$Q(x) \to$ n-βαθμού $n \leq m$.

$a > b$

$a, b \in \mathbb{N} \Rightarrow \exists\, \Pi(x), \sqrt{(x)}$

$a = b\pi + u \qquad P(x) = Q(x) \cdot \Pi(x) + \sqrt{(x)}$

$0 \leq u < b \qquad 0 \leq \deg(U(x)) < \deg(q(x))$

$\pi \in \mathbb{N} \qquad m \geq n$.

$$\int \frac{P(x)}{Q(x)}dx = \int \left(\Pi(x) + \frac{\sqrt{(x)}}{q(x)}\right)dx$$

$$\int \frac{P(x)}{Q(x)}dx \qquad P(x) \to m, \quad Q(x) \to n \qquad m < n.$$

$x^2 + x + 1$

Πρόταση κάθε πολυώνυμο $q(x)$ γράφεται ως εξής:
(με μεγιστοβάθμιο συντ. 1)

$$Q(x) = (x-a_1)^{r_1}(x-a_2)^{r_2}\cdots(x-a_k)^{r_k}(x^2+\beta_1 x+\gamma_1)^{s_1}\cdots(x^2+\beta_\ell x+\gamma_\ell)^{s_\ell}$$

$\Delta < 0 \qquad \Delta < 0$

$\int \frac{P(x)}{Q(x)}dx$, Πρόταση: Αν $\deg P(x) < \deg Q(x)$, $q(x)$ της μορφής (1).

$$\frac{P(x)}{Q(x)} = \frac{A_{11}}{x-a_1} + \frac{A_{12}}{(x-a_1)^2} + \cdots + \frac{A_{1r_1}}{(x-a_1)^{r_1}}$$

$$+ \left[\frac{A_{21}}{x-a_2} + \frac{A_{22}}{(x-a_2)^2} + \cdots + \frac{A_{2r_2}}{(x-a_2)^{r_2}}\right] + \cdots + \left[\frac{A_{k1}}{x-a_k} + \cdots + \frac{A_{kr_k}}{(x-a_k)^{r_k}}\right] +$$

$$+\left[\frac{B_{11}x+\Gamma_{11}}{(x^2+\beta_1 x+\gamma_1)}+\dots+\frac{B_{1s}x+\Gamma_{1s_1}}{(x^2+\beta_1 x+\delta_1)^{s_1}}\right]+\dots+\left[\frac{B_{e1}x+\Gamma_{e1}}{(x^2+\beta_e x+\delta_e)}+\dots\right.$$

$$+\frac{B_{es_e}x+\Gamma_{es_e}}{(x^2+\beta_e x+\delta_e)^{s_e}}$$

$$\frac{P(x)}{(x-1)(x-2)^2(x^2+2x+4)^2}=\frac{A_{11}}{x-1}+\frac{A_{21}}{x-2}+\frac{A_{22}}{(x-2)^2}+\frac{B_1x+\Gamma_1}{x^2+2x+4}$$

$$+\frac{B_2x+\Gamma_2}{(x^2+2x+4)^2}$$

$$\frac{x^2+3x+4}{(x-1)(x-2)^2(x+1)^4}=\frac{a_1}{x-1}+\frac{a_2}{x-2}+\frac{a_3}{(x-2)^2}+\frac{a_4}{x+1}+\frac{a_5}{(x+1)^2}+\frac{a_6}{(x+1)^3}$$

$$+\frac{a_7}{(x+1)^4}$$

$$\frac{3x^2+6}{x(x-1)(x+2)}$$

$$\int \underbrace{\boxed{\frac{3x^2+6}{x(x-1)(x+2)}}}_{f(x)}dx \quad ; f(x)=\frac{a}{x}+\frac{\beta}{x-1}+\frac{\gamma}{x+2} \Rightarrow$$

$$\int f(x)\,dx\dots$$

$$\frac{3x^2+6}{x(x-1)(x+2)} \Leftrightarrow 3x^2+6=a(x-1)(x+2)+\beta x(x+2)+\gamma x(x-1)$$

$$\left.\begin{aligned}a+b+c&=3\\2b+a-c&=0\\-2a&=6\end{aligned}\right\}\quad a=-3 \quad b=c=3$$

$$\int\frac{5x^2+12x+1}{x^3+3x^2-4}dx=\int\frac{5x^2+12x+1}{(x-1)^2(x+2)}dx$$

$$\int\frac{5x^2+12x+1}{(x-1)^2(x+2)}=\int f(x)=\int\left(\frac{a}{x-1}+\frac{\beta}{(x-1)^2}+\frac{\gamma}{x+2}\right)$$

$a+b=5$

$4a+b+c=12 \quad 4a-2b-c=1$

$a=2,\ b=3,\ \overset{c}{\overset{\shortparallel}{\gamma}}=1$

$\int f\,dx = 2\ln|x-1| + \int \frac{3}{(x-1)^2}\,dx + \gamma \ln|x+2|$

$2\ln|x-1| - \frac{3}{(x-1)} + \ln|x+2| + C$

$$\int \frac{x+1}{\underbrace{x^5-x^4+2x^3-2x^2+(x-1)}_{Q(x)}}\,dx =$$

$Q(x) = x^4(x-1)+2x^2(x-1)+(x-1) = (x^4+2x^2+1)(x-1) =$
$(x-1)(x^2+1)^2$

$$\frac{x+1}{(x-1)(x^2+1)^2} = \frac{a}{x-1} + \frac{\beta x+\gamma}{x^2+1} + \frac{\delta x+\epsilon}{(x^2+1)^2} \quad (1)$$

$(1) \Leftrightarrow x+1 = a(x^2+1)^2 + (\beta x+\gamma)(x-1)(x^2+1) + (\delta x+\epsilon)(x-1)$

$a+\beta=0 \qquad -\beta+\gamma+\epsilon-\delta=(1)$

$2a+\beta-\gamma+\delta=0 \qquad a-\gamma-\epsilon=(1)$

$a=1/2,\ \beta=-1/2=\gamma \qquad \int \frac{a}{x-1} = \checkmark$

$\delta=-1,\ \epsilon=0$

$\int \frac{-1/2x-1/2}{x^2+1},\ \int \frac{-x}{(x^2+1)^2}\,dx$

$$\int \frac{Bx+\Gamma}{(x^2+Bx+\gamma)^k}\,dx = \int \frac{(x^2+bx+\gamma)'}{(x^2+bx+\gamma)^k}\,dx = \frac{(x^2+bx+\gamma)^{-k+1}}{-k+1}+C$$

$$\int \frac{Bx+\Gamma}{(x^2+bx+\gamma)^k}dx \overset{k\in N}{=} \int \frac{\frac{B}{2}(2x+b)+\Gamma-\frac{Bb}{2}}{(x^2+bx+\gamma)^k}dx = I+J$$

$$I = \frac{B}{2}\int \frac{2x+b}{(x^2+bx+\gamma)^k}dx = \text{υπολογ εύκολα}$$

$$J = \left(\Gamma-\frac{Bb}{2}\right)\int \frac{dx}{(x^2+bx+\gamma)^k} \qquad \underbrace{\left(x+\frac{b}{2}\right)^2+p^2}_{\left(x+\frac{b}{2}\right)^2+\frac{4\gamma-b^2}{4}} = p^2\left[\left(\frac{x+\frac{b}{2}}{p}\right)^2+1\right]$$

$$\Delta = b^2-4\gamma<0$$

$$x^2+bx+\gamma = x^2+2\frac{b}{2}x+\frac{b^2}{4}+\left(\gamma-\frac{b^2}{4}\right).$$

$$x^2+x+1 = x^2+2\cdot\frac{1}{2}x+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1 = \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$$

$$= \left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2 = \left(\frac{\sqrt{3}}{2}\right)^2\left[\frac{\left(x+\frac{1}{2}\right)^2}{(\sqrt{3}/2)^2}+1\right]$$

$$\int \frac{dx}{(x^2+bx+\gamma)^k} = \int \frac{dx}{\left[p^2\left(\frac{\left(x+\frac{b}{2}\right)^2}{p^2}+1\right)\right]^k} = \frac{1}{p^{2k}}\int \frac{dx}{\left[\frac{\left(x+\frac{b}{2}\right)^2}{p^2}+1\right]^k}$$

$$= \frac{1}{p^{2k}}\int \frac{p\,dy}{(y^2+1)^k} = A\int \frac{dy}{(y^2+1)^k}$$

$$I_k = \int \frac{dy}{(y^2+1)^k} = \int y'\frac{1}{(y^2+1)^k}dy = y\frac{1}{(y^2+1)^k} - \int y\cdot\left(\frac{1}{(y^2+1)^k}\right)'dy$$

$$\left[(y^2+1)^{-k}\right]'$$

$k=1,2,\dots$ $\quad k>1$

$$k=1: \int \frac{dy}{y^2+1} = \text{τοξεφ}(y)+C$$

$\Delta<0$

$$= \frac{y}{(y^2+1)^k} - \int y(-k)(2y)\frac{1}{(y^2+1)^{k+1}}dy$$

$$= \frac{y}{(y^2+1)^k} + 2k\int \frac{y^2}{(y^2+1)^{k+1}}dy$$

$$\Rightarrow I_k = \frac{y}{(y^2+1)^k} + 2k\int \frac{(y^2+1)-1}{(y^2+1)^{k+1}}\,dy$$

$$= \frac{y}{(y^2+1)^k} + 2k\int\frac{dy}{(y^2+1)^k} - 2k\int\frac{dy}{(y^2+1)^{k+1}} = \frac{y}{(y^2+1)^k} + 2kI_k - 2k$$

$$2kI_{k+1} \Rightarrow I_{k+1} = \frac{1}{2k}\frac{y}{(y^2+1)^k} + \frac{(2k-1)}{2k}I_k$$

$$I_2 = \frac{1}{2}\left(\frac{y}{y^2+1}\right) + \frac{1}{2}I_1$$

$$I_1 = \operatorname{arctg}(y)$$

$$f(x) = \frac{x+1}{(x-1)(x^2+1)^2} = \frac{a}{x-1} + \frac{bx+c}{x^2+1} + \frac{dx+e}{(x^2+1)^2}$$

$a = 1/2$
$b = -1/2$
$d = -1,\ e = 0$

$$\int f = I_1 + I_2 + I_3.$$

$$I_1 = \int\frac{1/2}{x-1}\,dx = \frac{1}{2}\ln|x-1|$$

$$I_2 = -\frac{1}{2}\int\frac{x+1}{x^2+1}\,dx = -\frac{1}{2}\int\frac{\frac{1}{2}(2x)+1}{x^2+1}\,dx$$

$$= -\frac{1}{4}\int\frac{(x^2+1)'}{x^2+1} - \frac{1}{2}\int\frac{dx}{x^2+1} = -\frac{1}{4}\ln(x^2+1) - \frac{1}{2}\left[\operatorname{arctg}(x)\right] + c$$

$$I_3 = -\frac{1}{2}\int\frac{2x}{(x^2+1)^2}\,dx = -\frac{1}{2}\int\frac{(x^2+1)'}{(x^2+1)^2}\,dx = \frac{1}{2}\frac{1}{x^2+1} + c$$

$$\cancel{\int\frac{dx}{(x^2+x+1)^2}} \quad \cancel{\int\frac{Ax+B}{x^2+x+1}\,dx} \quad \frac{1}{(x^2+x+1)^2} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2}$$

$$= x^2+x+1 = \left(x^2+2\frac{1}{2}x+\frac{1}{4}\right)+\frac{3}{4} = \left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2$$

$$\left(\frac{\sqrt{3}}{2}\right)^2\left[\frac{\left(x+\frac{1}{2}\right)^2}{\left(\frac{\sqrt{3}}{2}\right)^2}+1\right]$$

$$I = \int\frac{dx}{(x^2+x+1)^2} = \int\frac{dx}{\left(\frac{\sqrt{3}}{2}\right)^2\left[\frac{\left(x+\frac{1}{2}\right)^2}{\left(\frac{\sqrt{3}}{2}\right)^2}+1\right]^2} =$$

$\frac{x+1/2}{\sqrt{3}/2}=y$

$$= \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^4}\int \frac{\frac{\sqrt{3}}{2}\,dy}{(y^2+1)^2} = A\cdot\int \frac{dy}{(y^2+1)^2} = AI_2$$

$$\int \log(x+\sqrt{x})\,dx = x\log(x+\sqrt{x}) - \int x\cdot\frac{(x+\sqrt{x})'}{x+\sqrt{x}}\,dx =$$

$$= x\log(x+\sqrt{x}) - \underbrace{\int x\,\frac{(x+\sqrt{x})'}{(x+\sqrt{x})}\,dx}_{J} =$$

$$J = \int x\,\frac{\left(1+\frac{1}{2\sqrt{x}}\right)}{x+\sqrt{x}}\,dx = \int \frac{x+\frac{1}{2}\sqrt{x}}{x+\sqrt{x}}\,dx$$

$\sqrt{x}=u \quad u>0$

$x=u^2$

$dx = 2u\,du$

$x>0$
$$= 2\int \frac{u^2+u\frac{1}{2}}{u^2+u}\,u\,du = 2\int \frac{u^2+u/2}{u+1}\,du$$

$$= \int \frac{2u^2+u}{u+1}\,du = \int \frac{2\left((u+1)^2-2u-1\right)+u}{u+1}\,du =$$

$$2\int (u+1)\,du + \int \frac{u-4u-2}{u+1}\,du = \int \frac{-3(u+1)+1}{u+1}\,du$$

$$= -3\int du + \int \frac{du}{u+1}$$

$$\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$$

$x = u^6$

$dx = 6u^5\,du$

$$= \int \frac{6u^5}{u^3+u^2}\,du = \int \frac{6u^3}{u+1}\,du = 6\int \frac{[(u+1)-1]^3}{u+1}\,d$$

$$= \cancel{6\int v^3 - 3v^2}$$

$u = v-1$

$du = dv$

$$6\int \frac{(v-1)^3}{v}\,dv$$

$$= 6\int \frac{v^3-3v^2+3v-1}{v}\,dv$$

Ορσ: $f:[a,b]\to\mathbb{R}$

$x_0 \in [a,b]$: $\exists\ f^{(k)}(x_0)\ \forall k=0,1,2,\dots,n$.

Ορίζεται πολ/μο Taylor n : f τάξης n στο x_0 βαθμού $\leq n$

$$\underset{\substack{\| \\ T_n(x)}}{T_{n,f,x_0}(x)} = f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\dots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

πολ/μο του $x\in\mathbb{R}$.

$T_n'(x) = f'(x_0)+(x-x_0)P(x)$

$T_n(x_0)=f(x_0)$
$T_n'(x_0)=f'(x_0)$
$T_n''(x_0)=f''(x_0)$
$\dots$
$T_n^{(n)}(x_0)=f^{(n)}(x_0)$ (Σ)

$T_n^{(n+1)}(x)=0\ \forall x\in\mathbb{R}$

Πρόταση:
Το $T_{n,f,x_0}(x)$ είναι το μοναδικό πολ/μο βαθμού $\leq n$ ώστε να ισχύουν οι (Σ)

Πρόταση:
Αν a_k, $k=0,1,2,\dots,n$ $a_k\in\mathbb{R}$, και $x_0\in\mathbb{R}$
$\exists!$ Πολ/μο βαθμού $\leq n$: $P(x)$
$P^{(k)}(x_0)=a_k$

Απόδειξη:

$$P(x)=a_0+\frac{a_1}{1!}(x-x_0)+\dots+\frac{a_n}{n!}(x-x_0)^n$$

$\Rightarrow P^{k}(x_0)=a_k\ \ \forall k=0,1,\dots,n$

Έστω ότι $\exists\ q(x)$ ώστε $q^{(k)}(x_0)=a_k\ \ \forall k=0,\dots,n$
βαθμού $\leq n$.

Θα δ.ο $P(x)=q(x)\ \forall x$

$P_1(x)=P(x)-q(x)$

$P_1^{(k)}(x_0)=0$, $\forall k=0,1,2,\dots,n$

Θα δ.ο $P_1(x)=0\ \forall x$

Γράφουμε $P_1(x)=C_nx^n+\dots+C_1x+C_0=C_n(x-0)^n+\dots+C_1(x-0)+C_0$

$P_1(x+x_0)=P_2(x)$

$P_2(0) = P_1(x_0) = 0$

$P_2'(0) = P_1'(x_0) = 0$

$\dots$

$P_2^{(n)}(0) = P_1^{(n)}(x_0) = 0$

$P_2(x) = b_0 + b_1 x + \dots + b_n x^n$

$P_2^{(k)}(0) = 0, \quad \forall k = 0, 1, \dots, n$

$$\left.\begin{array}{l} P_2(0) = b_0 = 0 \\ P_2'(0) = b_1 = 0 \\ P_2''(0) = 2b_2 = 0 \\ \dots \\ 0 = P_2^{(n)}(0) = n! b_n \end{array}\right\} \Rightarrow b_i = 0 \Rightarrow P_2 = 0 \Rightarrow P_1(x) = 0 \Rightarrow P(x) = q(x)$$

$f : [a,b] \to \mathbb{R}$

n-φορές παρ/μη στο x_0 $\exists f'(x_0), \dots, f^{(n)}(x_0)$

Θεωρούμε το υπόλοιπο Taylor της f στο x_0 τάξης n.

$R_{n,f,x_0}(x) = f(x) - T_{n,f,x_0}(x)$

$x \in [a,b]$

$f(x) = 2 + 3x - 7x^2 + 4x^3 + 5x^4 + 6x^5 = \sum_{k=0}^{5} a_k x^k$

$x_0 = 0$

Πολ/μο Taylor βαθμού $n=3$, $x_0 = 0$

$f(0) = a_0 = 2$

$f'(0) = 3 = a_1 \cdot 1!$

$f''(0) = 2! a_2$

$f^{(3)}(0) = 3! a_3$

$$T_{3,f,0}(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f^{(3)}(0)}{3!}(x-0)^3 = 2 + 3x - 7x^2 + 4x^3$$

$R_{3,f,0}(x) = 5x^4 + 6x^5 \quad \Big| \quad \dfrac{R_{3,f,0}(x)}{(x-0)^3} = 5x + 6x^2 \to 0$

Πρόταση: $f[a,b] \to \mathbb{R}$

$\exists\, f^{(n-1)}(x) \ \forall x \in [a,b]$.

$\exists\, f^{(n)}(x_0) \Rightarrow \lim\limits_{x\to x_0} \dfrac{R_{n,f,x_0}(x)}{(x-x_0)^n} = \lim\limits_{x\to x_0} \dfrac{f(x) - T_{n,f,x_0}(x)}{(x-x_0)^n}$

$= 0$.

$\exists\, f'(x_0) \quad \lim\limits_{x\to x_0} \dfrac{f(x) - T_{1,f,x_0}(x)}{x - x_0} = 0$

$T_{1,f,x_0}(x) = f(x_0) + \dfrac{f'(x_0)}{1!}(x - x_0)$

$\dfrac{f(x) - T_{1,f,x_0}(x)}{x - x_0} = \dfrac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \xrightarrow{x\to x_0} 0$

Λήμμα: Έστω $P(x)$ πολ/μο βαθμού $\leq n$ ώστε $\lim\limits_{x\to x_0} \dfrac{P(x)}{(x-x_0)^n} = 0$

Τότε $P(x) = 0 \ \forall x$

Απόδ. Με επαγωγή

$n = 1 \quad P(x) = ax + \beta$

και $\lim\limits_{x\to x_0} \dfrac{P(x)}{(x-x_0)} = 0 \Rightarrow P = 0$

$P(x_0) = 0$

$P(x_0) = 0 \Rightarrow x_0$ ρίζα του

$P(x) = 0$

$c(x - x_0) = 0 \cdot (x - x_0) = P(x) = 0 \ \forall x$

Αν ισχύει το λήμμα για τον φυσικό $n \in \mathbb{N}$

$n+1$ Υποθ. ότι $P(x)$ βαθμού $\leq n+1$

και ότι $\boxed{\lim\limits_{x\to x_0} \dfrac{P(x)}{(x-x_0)^{n+1}} = 0} \Rightarrow P = 0$

(1)

(1) $\Rightarrow \lim_{x \to x_0} P(x) = 0$ Αλλά $\lim_{x \to x_0} P(x) = P(x_0) \Rightarrow P(x_0) = 0$ (2)

(2) $\Rightarrow \exists q(x)$ $P(x) = (x - x_0)\, q(x)$

$\deg(q(x)) \leq n$

(1) $\Rightarrow \lim_{x} \dfrac{q(x)}{(x-x_0)^n} = 0 \xRightarrow[\text{επαγ}]{q=0} p = 0$

$f(x) = \dfrac{1}{1-x}$ $x \in (-1,1)$ $x_0 = 0$ Ψάχνω $T_{n,f,0}(x)$

$f'(x) = \dfrac{1}{(1-x)^2} = (1-x)^{-2}$

$f''(x) = \dfrac{2}{(1-x)^3}$, $f^{(k)}(x) = \dfrac{k!}{(1-x)^{k+1}}$ $\forall x \in (-1,1)$

$f'''(x) = \dfrac{3!}{(1-x)^4}$ $\forall k, \forall x$

$T_{n,f,0}(x) = f(0) + \dfrac{f'(0)}{1!}x + \dfrac{f''(0)}{2!}x^2 + \dots + \dfrac{f^{(n)}(0)}{n!}x^n$

$f(0) = 1$

$f^{(k)}(0) = k!$

$T_{n,f,0}(x) = 1 + x + \dots + x^n = \dfrac{x^{n+1} - 1}{x-1} \to \dfrac{-1}{x-1} = \dfrac{1}{1-x}$

$\lim_{n} T_{n,f,0}(x) = \lim_{n \to 0} \sum_{k=0}^{n} x^k = \sum_{k=0}^{\infty} x^k = \dfrac{1}{1-x}$

$\lim_{x \to 0} \dfrac{R_{n,f,0}(x)}{(x-0)^n} = 0$ $R_{n,f,0}(x) = f(x) - (1 + x \dots + x^n)$

$= \sum_{k=0}^{\infty} x^k - \sum_{k=0}^{n} x^k = \sum_{k=n+1}^{\infty} x^k =$

$x^{n+2} \sum_{k=0}^{\infty} x^k$

$\sum_{k=n+1}^{\infty} x^k = (x^{n+1} + x^{n+2}, \dots) = x^{n+1}(1 + x + x^2 + \dots) = x^{n+1}/1-x$

$\lim_{x \to 0} \dfrac{x}{1-x} = \lim_{x \to 0} \dfrac{x^{n+1}/1-x}{x^n}$

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Θεώρημα Taylor:

Δίνεται $f:[a,b]\to\mathbb{R}$: $\exists\, f^{(n+1)}(x)$ $\forall x\in[a,b]$ και $x_0\in[a,b]$. Αν επιπλέον $f^{(n+1)}$ είναι ολοκλ. στο $[a,b]$ τότε

$$P_{n,f,x_0}(x)=\frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)\,(x-t)^n\,dt,\quad \forall x\in[a,b]$$

Παρατήρηση: Ας υποθ. έχουμε επιπλέον $f^{(n+1)}$ συνεχ στο $[a,b]$

$R_{n,f,x_0}(x)=$

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g,\quad \exists\,\xi\in[a,b]$$

f συνεχ στο $[a,b]$, g ολοκληρ. στο $[a,b]$ διατηρεί πρόσημο στο $[a,b]$

$\boxed{\text{ΘΜΤ ολοκλ λογ.}}$

$$f(x)-\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

$$R_{n,f,x_0}(x)=\frac{1}{n!}f^{(n+1)}(\xi)\int_{x_0}^{x}(x-t)^n\,dt$$

$$=\frac{1}{n!}f^{(n+1)}(\xi)\cdot\frac{1}{n+1}(x-x_0)^{n+1}=\frac{1}{(n+1)!}f^{(n+1)}(\xi)(x-x_0)^{n+1}$$

για ξ μεταξύ x_0, x

$f(x)=e^x$

$x\in\mathbb{R}$

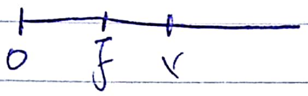
$x_0=0$.

$$R_{n,f,0}(x)=\frac{1}{n!}\int_0^x e^t(x-t)^n\,dt=\frac{e^\xi}{n!}\int_0^x (x-t)^n\,dt=\frac{x^{n+1}e^\xi}{n!}\le\frac{x^{n+1}e^x}{n!}$$

$\xrightarrow[n\to\infty]{?} 0$ για σταθ. x

ξ μεταξύ των $0,x$

$x>0$



$\xi=\xi(n,x)$

$0<\xi<x$

$x>0$

$$0\xleftarrow[n\to\infty]{x>0}\frac{x^{n+1}}{n!}=a_n,\quad a_n>0$$

$$\frac{a_{n+1}}{a_n}=\frac{\dfrac{x^{n+2}}{(n+1)!}}{\dfrac{x^{n+1}}{n!}}=\frac{x}{n+1}\to 0<1$$

$\sum a_n$, $a_n>0$, $\frac{a_{n+1}}{a_n}\to \ell<1 \Rightarrow \sum a_n$, $\ell\geq 0$

$\sum a_n$ συγκλ. $\Rightarrow a_n \to 0$

$a_n = S_n - S_{n-1} \to \ell - \ell = 0$

ii) $x<0$ $\quad R_{n,f,0}(x) = \frac{1}{n!} e^{\xi} x^{n+1} \Rightarrow |R_{n,f,0}(x)| = \frac{1}{n!} e^{\xi}(x)^{n+1} \to 0$

$$= e^{\xi} \frac{(|x|)^{n+1}}{n!} \leq \frac{|x|^{n+1}}{n!} \to 0 \qquad \xi<0 \Rightarrow e^{\xi}<1$$

Δ.Ο. $\lim R_{n,f,0}(x) = 0$

$f(x) = e^x \quad \forall x \in \mathbb{R}$ $\qquad \sum_{k=0}^{+\infty} \frac{x^k}{k!} = e^x \quad \forall x \in \mathbb{R}$

$x_0 = 0$

$$\Rightarrow \lim_{n\to\infty} \Big[\underbrace{f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k}_{0} \Big] \Rightarrow \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(x) = e^x \quad \forall x \in \mathbb{R}$$

$f: \mathbb{R} \to$ $\qquad \exists?\ g'(x) = ;$

συνεχ.

$\delta > 0 = 2$ $\qquad g(x) = \int_{x-\delta}^{x+\delta} f(t)\,dt$ $\quad \Big| \quad F_1(x) = \int_0^x F_1(t)\,dt$, $\quad F_1'(x) = f_1(x)$ $\quad \Big| \quad g(x) = \int_0^{x+\delta} f(t)\,dt + \int_0 f(t)\,dt$.

$$= \int_0^{x+\delta} f - \int_0^{x-\delta} f \to F(x+\delta) - F(x-\delta) = g(x)$$

$F'(x) = f(x)$

$F(x) = \int_0^x f(t)\,dt$

$f: [1,+\infty) \to \mathbb{R}$ συνεχ.

Ορίζουμε $\quad F(x) = \int_1^x f\left(\frac{x}{t}\right) dt$

για $x \geq 1$ $\qquad \exists F'$;

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$\frac{x}{t} = u \Rightarrow t = \frac{x}{u}$. $\quad dt = -\frac{x}{u^2}du$.

$$F(x) = \int_x^1 f(u)\left(-\frac{x}{u^2}\right)du = x\int_1^x \frac{f(u)}{u^2}du$$

$$F'(x) = x'\cdot\left(\int_1^x \frac{f(u)}{u^2}du\right) + x\left(\int_1^x \frac{f(u)}{u^2}du\right)'$$

$$= \int_1^x \frac{f(u)}{u^2}du + x\frac{f(x)}{x^2} = F(1) + \frac{f(x)}{x}$$

$f(x) = \cos x \quad x\in\mathbb{R}$.

$x_0 = 0$.

$$T_{n,f,x_0}(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$f'(x) = -\sin x$

$f''(x) = -\cos x$. $\qquad f^{(2k-1)}(x) = (-1)^k \sin x$.

$\forall x \quad f'''(x) = \sin x$. $\qquad f^{(2k)}(x) = (-1)^k\cos x$.

$f^4(x) = \cos x$. $\qquad \forall x\in\mathbb{R}$

$f^{(2k-1)}(0) = 0, \quad f^{(2k)}(0) = (-1)^k, \quad \forall k\dots$

$T_{2n,f,x_0}(x) \quad T_{2n,f,x_0} = 0$

$$(x) = f(x_0) + \frac{f''(0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(2n)}(0)}{2n!}x^{2n}$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \dots + \frac{(-1)^n}{2n!}x^{2n}$$

$$= \sum_{k=0}^{n}\frac{(-1)^k}{(2k)!}x^{2k}$$

$$R_{2n,f,0}(x) = f(x) - T_{2n,f,0}(x) = \frac{1}{(2n)!}\int_0^x f^{(2n+1)}(t)(x-t)^{2n}dt =$$

$$= \frac{1}{(2n)!}f^{(2n+1)}(\xi)\int_0^x (x-t)^{2n}dt = \frac{1}{(2n)!}f^{(2n+1)}(\xi)\frac{x^{2n+1}}{2n+1} = \frac{1}{(2n+1)!}f^{(2n+1)}(\xi)x^{2n+1}, \quad x>0$$

$x>0$

$\exists \xi$ μεταξύ των $0, x$

$$\frac{1}{(2n+1)!}\left((-1)^{n+1}\sin\xi\right)x^{2n+1} \qquad \xi=\xi(n,x)$$

$$|R_{2n,f,0}(x)| = \frac{|\sin\xi|\,x^{2n+1}}{(2n+1)!} \leq \underbrace{\frac{x^{2n+1}}{(2n+1)!}}_{a_n}$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{|x|^{2n+3}}{(2n+3)!}}{\frac{|x|^{2n+1}}{(2n+1)!}} = \frac{|x|^2}{(2n+2)(2n+3)} \to 0 \in [0,1)$$

$$\lim_{n\to\infty} R_{2n,f,0} = f(x) - \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k} = 0 \to f(x)=\cos x = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}$$

ASK) $f(x)=\sin(x)$.

$x\in\mathbb{R}$.

$f^{(2k)}(0)=0$.

$f^{(2k+1)}(0)=(-1)^k \ \forall k$

$$T_{2n+1,f,0}(x) = x-\frac{x^3}{3!}+\dots+\frac{(-1)^n x^{2n+1}}{(2n+1)!} \xrightarrow[R_{2n}\to 0]{n\to\infty} f(x) \ \forall x$$

$$R_{2n+1,f,0}(x) \underset{n\to\infty}{\to} 0 \quad (\text{ask})$$

$f(x)=\ln(1+x)$

$1>x\geq 0$, $x_0=0$

$f^{(k)}(0)$

$f^{(k)}(x)$

$x\in(-1,1)$

$$f'(x)=\frac{1}{1+x} \qquad f''(x)=\frac{(-1)}{(1+x)^2}$$

$$f'''(x)=\frac{2}{(1+x)^3}, \qquad f^{IV}(x)=\frac{-3!}{(1+x)^4}$$

$$f^{(k)}(x)=\frac{(-1)^{k-1}(k-1)!}{(1+x)^k}, \quad \forall x\in(-1,1).$$

$$f^{(k)}(0)=(-1)^{k-1}(k-1)!$$

$$T_{n,f,0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} \frac{(-1)^{k-1}}{k} x^k \qquad 1 > x \geq 0.$$

$$R_{n,f,0}(x) = \frac{1}{n!} f^{(n+1)}(\xi) x^{n+1}$$

$$0 \leq \xi < x \leq 1$$

$$= \frac{1}{n!} x^{n+1} \frac{(-1)^n \cdot n!}{(1+\xi)^n}$$

$$|R_{n,f,0}(x)| = \frac{x^{n+1}}{(1+\xi)^n} \leq x^{n+1}$$

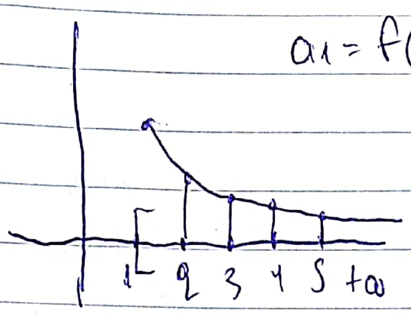
$$\downarrow$$

$$0$$

Κριτήριο (ολοκληρώματος).

$f: [1,+\infty) \to \mathbb{R}^+ \cup \{0\}$

f συνεχ., φθίνουσα, $\lim_{x\to+\infty} f(x) = 0$.



$a_n = f(n)$, $n = 1, 2, \ldots$ $\sum_{n=1}^{\infty} f(n)$ συγκλίνει?

$f(n) \geq 0$.

$\downarrow$

0

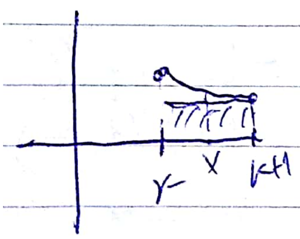
Τότε η σειρά $\sum_{n=1}^{\infty} f(n)$ συγκλίνει $\Leftrightarrow$

$\exists \lim_{n\to\infty} \int_1^n f(x)dx$ $\qquad \sum_{n=1}^{\infty} \frac{1}{n} = \sum f(n)$ δεν συγκλίνει $\Leftrightarrow \nexists \lim_{n\to\infty} \int_1^n f(x)dx$.

$f(x) = \frac{1}{x}$, $x \geq 1$ $\qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$, $f(x) = \frac{1}{x^2}$ $\qquad \lim_{n\to+\infty} \int_1^n \frac{dx}{x} = \lim_{n\to 0} \ln(n) = +\infty$

$\lim_n \int^n \frac{dx}{x^2}$ $\exists$ $\qquad \lim_n \left[-\frac{1}{x}\right]^n = \lim_n \left(-\frac{1}{n} + 1\right) = 1$

$k \in \mathbb{N}$ $\qquad k \leq x \leq k+1 \to f(k+1) \leq f(x) \leq f(k)$.



$$\int_k^{k+1} f(k+1)dx \leq \int_k^{k+1} f(x)dx \leq \int_k^{k+1} f(x)dx$$

$$\parallel$$

$$f(k+1) \leq \int_k^{k+1} f(x)dx \leq f(k) \; ①$$

$$S_n = \sum_{k=1}^{n} f(k)$$

$$\overset{n\geq 1}{\Rightarrow} \sum_{k=1}^{n} f(k+1) \leq \underbrace{\sum_{k=1}^{n} \int_{k}^{k+1} f(x)dx}_{\overset{||}{n+1}} \leq \sum_{k=1}^{n} f(k)$$

$$S_{n+1} - f(1) \leq \int_{1}^{n+1} f(x)dx \leq S_n$$

Έστω ότι $\sum_{k=1}^{\infty} f(k)$ συγκλίνει

$(S_n)_n \qquad S_n \leq S_{n+1} = S_n + f(n+1)$

$$\exists \lim_{n} S_n \in \mathbb{R} \Rightarrow S_n \leq \lim_{n} S_n \underset{}{=} \sum_{k=1}^{\infty} f(k)$$

$$\lim \delta_n = \lim \int_{1}^{n} |f(x)dx| \quad \exists (?)$$

$$\delta_n = \int_{1}^{n} f(x)dx \qquad \delta_{n+1} = \int_{1}^{n+1} f(x)dx$$

$\sum_{k=1}^{\infty} f(k)$ συγκλίνει $\Rightarrow S_n \leq M \ \forall n$ για κάποιο $M>0$.

$$\Rightarrow \delta_{n+1} = \int_{1}^{n+1} f(x)dx \leq S_n \leq M \ \forall n.$$

$$\delta_{n+1} \leq M. \ \forall n.$$

$$\exists \lim_{n} \delta_n = \lim_{n} \int_{1}^{n} f(x)dx$$

$$0 \leq \delta_{n+1} - \delta_n = \int_{n}^{n+1} f(x)dx \overset{f(x)\geq 0}{\geq} 0\,(n+1-n) = 1 \cdot 0 = 0$$

$\exists \lim \delta_n = \ell.$

$\Rightarrow \ell \ \delta_n \leq \ell \Rightarrow \delta_{n+1} \leq \ell.$ Όμως $S_{n+1} \leq \delta_{n+1} + f(1) \leq \ell + f(1) \Rightarrow$

$$\sum_{k=1}^{n+1} f(k) \leq \ell + f(1) \Rightarrow \exists \lim S_n = \sum_{k=1}^{n} f(k) \qquad S_n \leq S_{n+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)} \quad a_n. \qquad \text{συγκλίνει;}$$

$$\sum_{n=1}^{\infty} f(n) \text{ συγκλίνει } \Leftrightarrow.$$

$\exists f(x)$ ώστε $f(n)=a_n$.
$x \geq 1$

$$\exists \lim_n \int_1^n \frac{(\ln(x+1))'}{\ln(x+1)}\,dx$$

$$f(x) = \frac{1}{(x+1)\ln(x+1)}$$

$$\parallel$$

$$\exists \lim_n \left[\ln(\ln(x+1))\right]_1^n$$

$$\parallel$$

$$\exists \lim_n \left[\ln(\ln(n+1)) - \ln(\ln 2)\right]$$

$$\parallel$$

$$+\infty$$